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Nonlinear scattering processes in the presence of a quantised radiation field: I. Non-relativistic treatment

J Bergou and S Varró

Central Research Institute for Physics H-1525 Budapest, POB 49, Hungary

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Abstract. The Hamiltonian of a non-relativistic free charged particle interacting with one quantised mode of the electromagnetic field is diagonalised exactly in dipole approximation. The diagonalisation is simpler for a circularly polarised mode, but it is shown that the case of a linearly polarised mode can be reduced to the circularly polarised one. Exact stationary and time-dependent states of this Hamiltonian are then calculated. Based on these results, the rederivation of the nonlinear inverse and direct bremsstrahlung's cross section is presented and the connection with earlier (semiclassical) approaches is also established. Photon number distribution functions for different initial conditions are also determined in analytical form.

1. Introduction

This work is devoted to investigating an important model for the interaction between a free electron and an optical plane electromagnetic wave. In the model the electron is described by non-relativistic quantum mechanics, and the vector potential specifying the mode of the radiation field to be dealt with is considered as a quantised field quantity.

In the past fifteen years various nonlinear interaction processes of free charged particles with light (such as intense Compton scattering, direct and inverse bremsstrahlung in the presence of an intense light field, etc) have been very carefully investigated by semiclassical methods (Ehlotzky 1978, Kroll and Watson 1973). The idea common to these methods was to use the exact wavefunctions of the electron in an external plane electromagnetic wave as a basis set when treating the other interactions (with a new mode or with a background potential) with the help of perturbation theory. In spite of the fact that in calculations of this kind light is dealt with as a classical field, transition amplitudes of the electron and energy–momentum balance of the scattering processes can most easily be interpreted in terms of the photon picture. Thus, the approach to these problems proposed in the present paper was motivated in one respect by the necessity for the foundation of the above-mentioned intuitive photon picture. On the other hand, it is also clear that the external field approximation used in the semiclassical theory for the photon field becomes inapplicable if one wishes to consider absorption of light by free electrons (low-density electron gas). This holds in particular when the depletion of an optical mode due to nonlinear inverse bremsstrahlung of free electrons is considered.

In view of this, we consider that a non-perturbative description beyond the usual semiclassical treatment of the multiphoton transitions of a free electron is likely to be of

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interest. The starting point of our method is the exact solution to the Schrödinger equation of the system 'electron + quantised electromagnetic mode'; therefore the next two sections are devoted to finding the stationary and time-dependent states of this system. In the following section the solutions obtained in this way are applied to the description of the nonlinear bremsstrahlung of a free electron, and we rederive by simple methods the cross section of the process determined a few years ago by other methods (Osborn 1972). In § 5 the aforementioned photon statistics are investigated and in the last section the results are briefly summarised and discussed.

2. Exact solution for the case of a circularly polarised mode

In this section we solve exactly the Schrödinger equation of the system 'one electron + one circularly polarised quantised mode of the EM field'. From the beginning we introduce dipole approximation, and the vector potential reads

$$\boldsymbol{A} = \alpha(\boldsymbol{\varepsilon} a + \boldsymbol{\varepsilon}^* a^+) \tag{2.1}$$

where

$$\alpha = c(2\pi\hbar/\omega V)^{1/2} \tag{2.1a}$$

and

$$\boldsymbol{\varepsilon}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*\boldsymbol{\varepsilon}^* = 0, \qquad \boldsymbol{\varepsilon}^*\boldsymbol{\varepsilon} = 1. \tag{2.1b}$$

Here a and a^+ are the absorption and emission operators belonging to the mode specified by the $\boldsymbol{\varepsilon}$ complex polarisation vector, V is the volume of the quantisation box and ω is the mode frequency. The symbols * and $^+$ stand for complex and Hermitian conjugation, respectively.

In general, the interaction of radiation with free charges is described by the following Schrödinger equation in the non-relativistic case:

$$\mathrm{i}\hbar\partial\psi(t)/\partial t = H\psi(t) \tag{2.2}$$

where the Hamiltonian of the system can be decomposed into three parts in a natural way:

$$H = H_\mathrm{e} + H_\mathrm{f} + H_\mathrm{I}, \tag{2.2a}$$

where

$$H_\mathrm{e} = \frac{\boldsymbol{p}^2}{2m}, \qquad H_\mathrm{f} = \hbar\omega(a^+a + \tfrac{1}{2}), \qquad H_\mathrm{I} = -\frac{e}{mc}\boldsymbol{A}\boldsymbol{p} + \frac{e^2}{2mc^2}\boldsymbol{A}^2. \tag{2.2b}$$

In the case of a circularly polarised mode, the $\boldsymbol{A}^2$ term of the H_I interaction Hamiltonian can be incorporated into H_f, the Hamiltonian of the field. By using equations (2.1a) and (2.1b), we obtain

$$H_\mathrm{f} + (e^2/2mc^2)\boldsymbol{A}^2 = \hbar\Omega(a^+a + \tfrac{1}{2}), \tag{2.3}$$

$$\Omega = \omega(1 + \omega_\mathrm{p}^2/2\omega^2), \qquad \omega_\mathrm{p}^2 = 4\pi e^2/mV. \tag{2.3a}$$

Here ω_p can be formally identified with the plasma frequency of an electron gas having a density of one electron/quantisation volume. By using (2.3) and (2.3a) we can see that

the total Hamiltonian of the full system has the form

$$H = \boldsymbol{p}^2/2m + \hbar\Omega(a^+a + \tfrac{1}{2}) - (e\alpha/mc)\boldsymbol{p}(\boldsymbol{\varepsilon}a + \boldsymbol{\varepsilon}^* a^+). \tag{2.4}$$

Let us turn our attention to the interaction still present in (2.4), which is now linear in a and a^+. As we will show, it can be eliminated by an appropriate unitary transformation. To perform this, let us introduce the unitary operator

$$D_\sigma = \exp(\sigma a^+ - \sigma^+ a) \tag{2.5}$$

which has the displacement property

$$D_\sigma^{-1} a D_\sigma = a + \sigma, \qquad D_\sigma^{-1} a^+ D_\sigma = a^+ + \sigma^+. \tag{2.5a}$$

The operator quantity σ will be specified later on; at present we merely assume that it commutes with a, a^+, $\boldsymbol{p}$, and σ^+. Perform now the unitary transformation on H:

$$\begin{aligned}\mathscr{H} = D_\sigma^{-1} H D_\sigma \\ = \frac{\boldsymbol{p}^2}{2m} + \hbar\Omega(a^+a + \tfrac{1}{2} + \sigma^+\sigma) + \left(\hbar\Omega\sigma^+ - \frac{e\alpha}{mc}\boldsymbol{p}\boldsymbol{\varepsilon}\right)a \\ + \left(\hbar\Omega\sigma - \frac{e\alpha}{mc}\boldsymbol{p}\boldsymbol{\varepsilon}^*\right)a^+ - \frac{e\alpha}{mc}\boldsymbol{p}(\boldsymbol{\varepsilon}\sigma + \boldsymbol{\varepsilon}^*\sigma^+).\end{aligned} \tag{2.6}$$

If we define σ by

$$\hbar\Omega\sigma = (e\alpha/mc)\boldsymbol{p}\boldsymbol{\varepsilon}^*, \tag{2.6a}$$

then the transformed Hamiltonian $\mathscr{H}$ does not contain any interaction term, and the operators acting on electron coordinates are entirely separate from those acting on photon coordinates. Furthermore, in the basis $\{|\boldsymbol{p}n\rangle\} = \{|\boldsymbol{p}\rangle|n\rangle\}$ it becomes diagonal (here $|\boldsymbol{p}\rangle$ is a momentum eigenstate of the electron and $|n\rangle$ is a number state (Fock state) of the field). It is also clear that the above method is suitable for eliminating the interaction with the long-wavelength part of the full radiation field if coupling between modes is neglected. The unitary transformation acting in this way is an infinite product of displacement operators of the type (2.5), belonging to one particular mode. A similar transformation was introduced by Bloch and Nordsieck (1937) when they first solved the problem of the elimination of infrared divergences from QED.

The stationary Schrödinger equation belonging to (2.2) is

$$H\varphi = E\varphi, \qquad \varphi = \exp[(\mathrm{i}/\hbar)Et]\psi. \tag{2.7}$$

Having performed the unitary transformation (2.5), we can see that this becomes equivalent to the eigenvalue equation

$$\mathscr{H}\chi = E\chi, \qquad \chi = D_\sigma^{-1}\varphi, \tag{2.8}$$

where, according to (2.6) and (2.6a),

$$\mathscr{H} = \boldsymbol{p}^2/2m + \hbar\Omega(a^+a + \tfrac{1}{2} - \sigma^+\sigma). \tag{2.8a}$$

χ can be chosen in the form of a product of the $|\boldsymbol{p}\rangle$ free electron momentum eigenstate (plane wave) and the $|n\rangle$ Fock state (number eigenstate) for the photons. The solutions of (2.7) can easily be obtained by the inverse transformation in the form

$$\varphi = \varphi_{\boldsymbol{p}n} = |\boldsymbol{p}\rangle D_\sigma |n\rangle, \qquad n = 0, 1, 2, \ldots. \tag{2.8b}$$

The corresponding energy eigenvalues are

$$E = E(\boldsymbol{p}, n) = \boldsymbol{p}^2/2m + \hbar\Omega(n + \tfrac{1}{2} - \sigma^*\sigma), \qquad n = 0, 1, 2, \ldots . \tag{2.8c}$$

It should be noted here that σ is already a c-number, defined by the matrix element of (2.6a) in momentum representation. Due to the unitarity of D_σ, the states φ_{pn} in equation (2.8b) form a complete orthonormal system. From equations (2.8b) and (2.8c), it can be seen that at a fixed value of the momentum the lowest energy state of the mode ($n = 0$) is a coherent state, since D_σ is the creation operator of the quantum mechanical coherent state with parameter σ (Glauber 1963a, b). Hence

$$\varphi_{p0} = |\boldsymbol{p}\rangle|\sigma\rangle, \qquad |\sigma\rangle = D_\sigma|0\rangle. \tag{2.9}$$

If in addition to stationary states one is also interested in the dynamics of the system, one has to solve directly the time-dependent equation (2.2). Let

$$\psi(t) = \exp\left[-(\mathrm{i}/\hbar)\hbar\Omega(a^+a + \tfrac{1}{2})t\right]\varphi(t). \tag{2.10}$$

In the interaction picture with respect to the free photon field defined by equation (2.10), equation (2.2) has the form

$$K(t)\varphi(t) = \mathrm{i}\hbar\partial\varphi(t)/\partial t \tag{2.11}$$

where

$$K(t) = \boldsymbol{p}^2/2m - (e\alpha/mc)\boldsymbol{p}(\boldsymbol{\varepsilon}a\ \mathrm{e}^{-\mathrm{i}\Omega t} + \boldsymbol{\varepsilon}^*a^+\ \mathrm{e}^{\mathrm{i}\Omega t}). \tag{2.11a}$$

Looking for the solution of (2.11) in the form

$$\varphi(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau[K(\tau) + f(\tau)]\right)\phi(t_0), \tag{2.12}$$

we obtain for the real c-number function $f(t)$ the expression

$$f(t) = -\frac{\mathrm{i}}{\hbar}\frac{1}{2}\left[K(t), \int_{t_0}^{t} K(\tau)\,\mathrm{d}\tau\right] = \hbar\Omega\sigma^*\sigma[\cos\Omega(t - t_0) - 1] \tag{2.12a}$$

where we used the relation $\partial\,\mathrm{e}^B/\partial t = \{\partial B/\partial t + \frac{1}{2}[B, \partial B/\partial t]\}\,\mathrm{e}^B$, if both B and $\partial B/\partial t$ commute with $[B, \partial B/\partial t]$.

Finally, the solution of equation (2.11) takes the form

$$\varphi(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{\boldsymbol{p}^2}{2m}(t - t_0) + \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right)\right]D[\sigma(t, t_0)]\varphi(t_0) \tag{2.13}$$

where

$$D[\sigma(t, t_0)] = \exp[\sigma(t, t_0)a^+ - \sigma^*(t, t_0)a] \tag{2.13a}$$

and

$$\sigma(t, t_0) = (e\alpha/mc\hbar\Omega)(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t_0}). \tag{2.13b}$$

If $\varphi(t_0)$ describes a bare electron propagating in vacuum, then the photonic part of $\varphi(t)$ corresponds to a coherent state of parameter $\sigma(t, t_0)$ given by equation (2.13b) (Glauber 1963b). We also note that in this case $\psi(t)$ is coherent too. Therefore one may conclude that the long-wavelength components of the self field of an electron are in a coherent state. We shall further investigate photon statistics with the help of the stationary and time-dependent solutions in § 5.

3. Solution for the case of linear polarisation

In the previous section we diagonalised the Hamiltonian of the system 'one electron + one circularly polarised mode' with the help of the displacement operator D_σ. This method leads to similar results in the case of a linearly polarised mode as well, provided that an appropriate transformation of the a and a^+ operators is introduced. This transformation reduces the problem to a problem identical in form with the circularly polarised case.

The Hamiltonian of the system now reads

$$H = \frac{\boldsymbol{p}^2}{2m} + \hbar\Omega(a^+a + \tfrac{1}{2}) - \frac{e\alpha}{mc}\boldsymbol{pe}(a + a^+) + \frac{e^2\alpha^2}{2mc^2}[a^2 + (a^+)^2]. \tag{3.1}$$

Here $\boldsymbol{e}$ is a real unit vector of polarisation, and a and a^+ are absorption and emission operators of the linearly polarised mode; Ω is again given by (2.3a).

Let b_1 and b_2 be real numbers and let us define the operators b, b^+ by the following relations.

$$b = b_1 a + b_2 a^+, \qquad b^+ = b_1 a^+ + b_2 a. \tag{3.2}$$

If

$$b_1^2 - b_2^2 = 1 \tag{3.2a}$$

then the commutation relation for b and b^+ is the same as for a and a^+,

$$[b, b^+] = 1. \tag{3.2b}$$

It is convenient at this stage to introduce a new parameter θ with the substitutions

$$b_1 = \cosh\theta, \qquad b_2 = \sinh\theta. \tag{3.2c}$$

Equation (3.2a) will then automatically be fulfilled and the transformed operators b and b^+ are now specified by a single real parameter.

It is easy to check that upon substitution of b and b^+ into (3.1) and requiring

$$\left(\frac{e^2\alpha^2}{mc^2}\Big/\hbar\Omega\right) = \tanh 2\theta \tag{3.3}$$

to hold, we obtain the Hamiltonian, in terms of b and b^+, in a form similar to the case of the circular polarisation, equation (2.4):

$$H = \boldsymbol{p}^2/2m + \hbar\nu(b^+b + \tfrac{1}{2}) - (e\alpha/mc)\boldsymbol{pe}\,\mathrm{e}^{-\Theta}(b + b^+) \tag{3.4}$$

where

$$\nu = \Omega\,\mathrm{sech}\,2\theta \tag{3.4a}$$

Furthermore, from equation (3.3) one can determine the actual value of θ, and the transformed frequency ν can also be expressed in terms of the radiation frequency ω and the plasma frequency ω_p:

$$\theta = \tfrac{1}{2}\tanh^{-1}\frac{\omega_\mathrm{p}^2}{2\omega^2(1 + \omega_\mathrm{p}^2/2\omega^2)} = \tfrac{1}{2}\tanh^{-1}\frac{\beta}{1+\beta} \tag{3.4b}$$

and

$$\nu = \omega(1+\beta)\{1 - [\beta/(1+\beta)]^2\}^{1/2} \tag{3.4c}$$

where

$$\beta = \omega_p^2/2\omega^2. \tag{3.4d}$$

In the case of $\omega_p \ll \omega$, β becomes very small and then ν and Ω are practically identical with ω.

The stationary and time-dependent Schrödinger equations belonging to the Hamiltonian (3.4) can be solved completely analogously using the method outlined in the previous section. With the help of the displacement operator

$$D_\rho = \exp[\rho(b^+ - b)], \qquad \rho = (e\alpha/mc\hbar\nu)\boldsymbol{pe}\ \mathrm{e}^{-\theta}, \tag{3.5}$$

one can diagonalise the Hamiltonian (3.4) in the Fock basis $\{|l\rangle_b\}$ of the b^+b operator:

$$[\boldsymbol{p}^2/2m + \hbar\nu(b^+b + \tfrac{1}{2} - \rho^2)]D_\rho^{-1}\varphi_b = E_b D_\rho^{-1}\varphi_b. \tag{3.6}$$

Thus the solution of the stationary Schrödinger equation for the case of a linearly polarised mode becomes

$$\varphi_b = |\boldsymbol{p}\rangle D_\rho |l\rangle_b, \qquad l = 0, 1, 2, \ldots, \tag{3.6a}$$

$$E_b = E_b(\boldsymbol{p}, l) = \boldsymbol{p}^2/2m + \hbar\nu(l + \tfrac{1}{2} - \rho^2), \qquad l = 0, 1, 2, \ldots. \tag{3.6b}$$

$|\boldsymbol{p}\rangle$ is again a free plane wave with momentum $\boldsymbol{p}$ of the electron. The transition from the $\{|k\rangle_a\}$ Fock basis of the original a^+a operator to the $\{|l\rangle_b\}$ Fock basis of the b^+b operator is effected by the transformation

$$|l\rangle_b = \sum_{k=0}^{\infty} |k\rangle_{aa}\langle k|l\rangle_b = \sum_{k=0}^{\infty} |k\rangle_a G_{kl}(\theta). \tag{3.7}$$

The problem of the determination of the matrix elements ${}_a\langle k|l\rangle_b$ can be reduced to the separate determination of the matrix elements of the b vacuum (defined in the usual way as $b|0\rangle_b = 0$, and a more complicated matrix element defined entirely between the $|k\rangle_a$ states:

$${}_a\langle k|l\rangle_b = \sum_{m=0}^{\infty} \frac{1}{(l!)^{1/2}}\, {}_a\langle k|(a^+\cosh\theta + a\sinh\theta)^l|2m\rangle_{aa}\langle 2m|0\rangle_b \tag{3.7a}$$

where from the definition of the state $|0\rangle_b$ it can easily be shown that

$${}_a\langle 2m|0\rangle_b = (-\tanh\theta)^m\left(\frac{1\times 3\times\ldots(2m-1)}{1\times 2\times\ldots\times 2m}\right)^{1/2} {}_a\langle 0|0\rangle_b, \qquad m = 1, 2, \ldots, \tag{3.7b}$$

and

$${}_a\langle 0|0\rangle_b = (\cosh\theta)^{-1/2}. \tag{3.7c}$$

This last relation is a consequence of the normalisation condition ${}_b\langle 0|0\rangle_b = 1$ of the b vacuum. An explicit calculation of the matrix elements $G_{kl}(\theta)$ was first given by Tanabe (1973). All those ${}_a\langle k|l\rangle_b$ matrix elements disappear for which $k + l =$ odd number. If the electron–photon system is in the lth excited state with energy $E_b(p, l)$ then, depending on whether l is odd or even, this state according to equation (3.7) can be expressed from the linear superposition of Fock states with odd or even number of photons respectively, with the help of the D_ρ operator (3.5), as indicated in (3.6a). In particular, the vacuum of b^+b is a linear superposition of infinitely many photon states, each containing an even number of photons. This is not surprising if one takes into account that the transformation (3.2) eliminates the interaction part proportional to $[a^2 + (a^+)^2]$.

For the sake of completeness we also mention that the Bogoliubov transformation (3.2) can be effected by a unitary operator C_θ. Taking into account (3.2c), this generator has the form (Tanabe 1973)

$$C_\theta = \exp[\tfrac{1}{2}\Theta(a^+a^+ - aa)]. \tag{3.8}$$

The transformation (3.2) can now be written formally as

$$C_\theta^{-1} a C_\theta = b, \qquad C_\theta^{-1} a^+ C_\theta = b^+. \tag{3.8a}$$

The transition from the complete set of eigenstates $\{|l\rangle_a\}$ of the a^+a operator to the complete set of eigenstates $\{|l\rangle_b\}$ of the b^+b operator can be performed with the same operator C_θ; thus it gives the representation-independent definition of equation (3.7) as

$$|l\rangle_b = C_\theta^{-1}|l\rangle_a \tag{3.8b}$$

or in other words

$${}_a\langle k|l\rangle_b = {}_a\langle k|C_\theta^{-1}|l\rangle_a. \tag{3.8c}$$

Finally, for the wavefunction we can replace equation (3.6a) by

$$\begin{aligned}\varphi_b &= |\boldsymbol{p}\rangle D_\rho^{(b)}|l\rangle_b = |\boldsymbol{p}\rangle D_\rho^{(b)} C_\theta^{-1}|l\rangle_a \\ &= |\boldsymbol{p}\rangle C_\theta^{-1} C_\theta D_\rho^{(b)} C_\theta^{-1}|l\rangle_a = |\boldsymbol{p}\rangle C_\theta^{-1} D_\rho^{(a)}|l\rangle_a\end{aligned} \tag{3.9}$$

where C_θ and $D_\rho^{(\alpha)}$ are both now defined in the original Fock basis $\{|l\rangle_a\}$.

4. Application to direct and inverse bremsstrahlung

In this section we shall consider the scattering of electrons by a background potential in plasmas. We also allow the electron to interact with one quantised mode of the electromagnetic field during the scattering. This problem was first studied by Osborn (1972), who calculated the cross section of processes of this kind for the case with Coulomb-type background potential; with the help of cross sections he then determined energy absorption coefficients resulting from nonlinear direct and inverse bremsstrahlung. However, the method proposed by him is too tedious, and moreover the result given in that paper for the transition amplitudes is erroneous. From these amplitudes he determined the correct cross section formulae; nevertheless, the physical picture involved in his formulae is slightly different from that taken as a starting point for his calculations. This is why we feel it is necessary to reinvestigate the problem in some detail. For the sake of simplifying the following calculations we introduce the same assumptions as were made by Osborn. First, we consider the ionic background as Debye shielded and the reaction of electrons to the background is neglected, i.e. we assume infinitely massive ions. Second, as the Debye length is usually less than the wavelength of optical photons, this justifies the treatment of the radiation mode in dipole approximation ($e^{i\boldsymbol{kr}} \approx e^{-i\boldsymbol{kr}} \approx 1$).

The average number of photons in intense laser fields can be so large that interaction of the electrons with the field becomes stronger than the interaction with the ionic background. Therefore, we treat the $V(\boldsymbol{r})$ scattering potential as a perturbation (rather than using the proper continuum eigenfunctions in a central field). A detailed explanation of the method using semiclassical treatment is given in Bergou (1980).

In accordance with these preliminary considerations, we shall describe the initial and final states of the 'electron + one mode' system by the (2.8*a*)- or (3.6*b*)-type stationary wavefunctions specified by the $\boldsymbol{p}_i$, η and $\boldsymbol{p}_f$, $\eta \pm n$ initial and final momenta and photon numbers, respectively. The transition amplitude between such states therefore has the form (to first order in $V(\boldsymbol{r})$):

$$T_{fi}^{(\pm n)} = -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \exp[(i/\hbar)(E_f - E_i)t]\langle \boldsymbol{p}_f|V(r)|\boldsymbol{p}_i\rangle\langle \eta \pm n|D_{-\sigma_f}D_{\sigma_i}|\eta\rangle. \tag{4.1}$$

If we carry out time integration, we obtain the expression

$$T_{fi}^{(\pm n)} = -2\pi i\delta[E(\boldsymbol{p}_f, \eta \pm n) - E(\boldsymbol{p}_i, \eta)]V(\boldsymbol{Q})\langle \eta \pm n|D_{-\sigma_f}D_{\sigma_i}|\eta\rangle \tag{4.2}$$

where

$$V(\boldsymbol{Q}) = \int d^3r\, V(\boldsymbol{r})\, e^{-(i/\hbar)\boldsymbol{Q}\boldsymbol{r}}, \qquad \boldsymbol{Q} = \boldsymbol{p}_f - \boldsymbol{p}_i. \tag{4.2a}$$

To calculate the matrix elements more explicitly, we make use of the relations

$$D_{\sigma_1}D_{\sigma_2} = \exp[-\tfrac{1}{2}(\sigma_1\sigma_2^* - \sigma_1^*\sigma_2)]D_{\sigma_1+\sigma_2} \tag{4.3}$$

and

$$\langle l|D_\sigma|m\rangle = \exp(-\tfrac{1}{2}|\sigma|^2)(l!\, m!)^{1/2}\sum_{k=0}^{\min(l,m)} \frac{\sigma^{l-k}(-\sigma^*)^{m-k}}{(l-k)!(m-k)!k!}. \tag{4.4}$$

By using equations (4.3) and (4.4) in (4.2), we can write the matrix elements $\langle \eta \pm n|D_{-\sigma_f}D_{\sigma_i}|\eta\rangle$ as

$$\langle \eta \pm n|D_{-\sigma_f}D_{\sigma_i}|\eta\rangle = (\pm)\, e^{\pm in\chi}[(\eta \pm n)!\eta!]^{1/2}\exp[\tfrac{1}{2}(\sigma_f\sigma_i^* - \sigma_f^*\sigma_i]$$
$$\times \sum_{k=0}^{\min(\eta\pm n,\eta)} \frac{(-1)^k|\tau|^{2k+n}\exp(-\frac{1}{2}|\tau|^2)}{(n+k)!k![\min(\eta \pm n, \eta) - k]!} \tag{4.5}$$

where

$$\tau = \sigma_i - \sigma_f, \qquad \chi = \arg\tau. \tag{4.5a}$$

$(\pm)$ in front of the RHS of equation (4.5) should be replaced by $(-1)^n$ if the final state is $|\eta - n\rangle$, and by 1 if the final state is $|\eta + n\rangle$. The relation of these matrix elements to the generalised Laguerre polynomials is (Abramowitz and Stegun 1964)

$$\sum_{k=0}^{\eta} \frac{(-|\tau|^2)^k}{(n+k)!k!(\eta-k)!} = \frac{1}{(\eta+n)!}L_\eta^{(n)}(|\tau|^2) \tag{4.5b}$$

and

$$\sum_{k=0}^{\eta-n} \frac{(-|\tau|^2)^k}{(n+k)!k!(\eta-n-k)!} = \frac{1}{\eta!}L_{\eta-n}^{(n)}(|\tau|^2). \tag{4.5c}$$

Here we note that if one considers transitions between time-dependent states, one has to use (2.13)-type initial and final states. Hence

$$T_{fi} = -\frac{i}{\hbar}\int_{t_0}^{\infty} dt \exp\left[+\frac{i}{\hbar}\left(\frac{\boldsymbol{p}_f^2 - \boldsymbol{p}_i^2}{2m}(t-t_0) + \int_{t_0}^{t} d\tau(f_{p_f}(\tau) - f_{p_i}(\tau))\right)\right]$$
$$\times\langle \boldsymbol{p}_f|V(\boldsymbol{r})|\boldsymbol{p}_i\rangle\langle\gamma_f|D[-\sigma_f(t,t_0)]D[\sigma_i(t,t_0)]|\gamma_i\rangle. \tag{4.6}$$

Here $|\gamma_i\rangle$ and $|\gamma_f\rangle$ respectively denote arbitrary initial and final photon states. If, for example, $|\gamma_i\rangle = |\eta\rangle$ and $|\gamma_f\rangle = |\eta \pm n\rangle$ number states, then the photonic part in the integrand of (4.6) is entirely similarly to (4.5*a*), with the only exception that the corresponding σ's are now time dependent:

$$\langle \eta \pm n | D[-\sigma_f(t, t_0)] D[\sigma_i(t, t_0)] | \eta \rangle = (\pm)\, e^{\pm i n \chi(t)} [(\eta \pm n)!\eta!]^{1/2} \times \exp\{\tfrac{1}{2}[\sigma_f(t, t_0)\sigma_i^*(t, t_0) - \sigma_f^*(t, t_0)\sigma_i(t, t_0)]\} \times \sum_{k=0}^{\min(\eta \pm n, \eta)} \frac{(-1)^k |\tau(t)|^{2k+n} \exp[-\tfrac{1}{2}|\tau(t)|^2]}{(n+k)!k![\min(\eta \pm n, \eta) - k]!} \tag{4.7}$$

where

$$\tau(t) = \sigma_i(t, t_0) - \sigma_f(t, t_0), \qquad \chi(t) = \arg \tau(t). \tag{4.7a}$$

Here $\sigma_{i,f}(t, t_0)$ should be calculated according to (2.13*b*). Equation (4.7) clearly contradicts Osborn's corresponding result given by equation (7) of his paper (Osborn 1972), where the contribution from the photon part is given by an infinite sum. We note also that in his paper interaction with a linearly polarised mode was considered and the $\boldsymbol{A}^2$ term was arbitrarily dropped. This approximation is essentially identical with the circularly polarised case of the present paper.

We now turn our attention to the differences of transitions between stationary and time-dependent states. It is clear that the physical backgrounds of the matrix elements (4.1) and (4.6) are different. Equation (4.1) corresponds to transitions between two stationary states of the type (2.8*a*) (or (3.6*a*)). These states possess a well defined energy, and due to energy conservation (expressed by the Dirac delta function of equation (4.2))

$$E_f(\boldsymbol{p}_f, \eta \pm n) = E_i(\boldsymbol{p}_i, \eta), \tag{4.8}$$

which in the case of circular polarisation becomes

$$\boldsymbol{p}_f^2/2m - \hbar\Omega(|\sigma_f|^2 \mp n) = \boldsymbol{p}_i^2/2m - \hbar\Omega|\sigma_i|^2. \tag{4.8a}$$

Equation (4.8*a*) expresses the fact that during scattering the photon content changes. In § 5 we shall see that in the state $D_\sigma|\eta\rangle$ the average photon number is $\eta + |\sigma|^2$ and, since $|\sigma|^2$ is usually small for non-relativistic electrons, equation (4.8*a*) may be interpreted so that during scattering the electron emits or absorbs n photons. Of course, in the case of stationary states this does not mean that in the system initially there are η photons (since the initial state is not a number state), and this changes to $\eta \pm n$ during scattering.

In the following we evaluate the matrix element (4.5). The interesting case from the point of view of applications to real processes is the one when $\eta \gg n$. Then, by comparing it with the series expansion of the Bessel functions (Abramowitz and Stegun 1964),

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\tfrac{1}{2}z)^{2r+n}}{(n-r)!r!}, \qquad n \text{ integer},$$

we find that (4.5) can be well approximated by the formula

$$|\langle \eta \pm n | D_{-\sigma_f} D_{\sigma_i} | \eta \rangle|^2 \approx J_n^2(2|\tau|\sqrt{\eta}), \tag{4.9}$$

where for circular polarisation

$$|\tau| = (e\alpha/mc\hbar\Omega)|\boldsymbol{\varepsilon}\boldsymbol{Q}|. \tag{4.9a}$$

In classsical approximation, $\sqrt{\eta}$ in the argument of the Bessel function can be expressed through the amplitude of the vector potential for a classical field in the following way:

$$A_0 = 2\alpha\sqrt{\eta}, \qquad 2|\tau|\sqrt{\eta} = (eA_0/mc\hbar\Omega)|\boldsymbol{\varepsilon Q}|. \tag{4.10}$$

The scattering cross section of the nonlinear direct and inverse bremsstrahlung can now be obtained from the transition amplitude (4.2) in the usual way. After introducing the approximation indicated by (4.9) and (4.10), one finally obtains

$$\frac{d\sigma^{(\pm n)}}{d\Omega_f} = \frac{p_f}{p_i} J_n^2(z) \frac{d\sigma_{\text{Born}}}{d\Omega_f}, \qquad z = \frac{eA_0}{mc\hbar\Omega}|\boldsymbol{\varepsilon Q}|, \tag{4.11}$$

where

$$d\sigma_{\text{Born}}/d\Omega_f = (m/2\pi\hbar^2)^2 |V(\boldsymbol{Q})|^2 \tag{4.11a}$$

is the cross section of the elastic scattering in Born approximation and p_f and p_i are related via the energy conservation law (4.8*a*). It can also be shown (see the fundamental work of Nordsieck (1937)) that the nth-order cross section of the low-frequency multiphoton direct and inverse bremsstrahlung has the more general structure

$$\frac{d\sigma^{(\pm n)}}{d\Omega_f} \approx \frac{p_f}{p_i} J_n^2(z) \frac{d\sigma_{\text{el}}}{d\Omega_f}. \tag{4.12}$$

Here $d\sigma_{\text{el}}/d\Omega_f$ is the full elastic cross section of the scattering by $V(\boldsymbol{r})$. Thus equation (4.12) reproduces the semiclassical result obtained in a different way (Kroll and Watson 1973).

The energy absorption coefficient of electrons (connected with the heating of plasma due to nonlinear inverse bremsstrahlung) was also computed by Osborn (1972). Since this involves a summation over different $n-s$ (n = number of absorbed and emitted photons), one has in principle to use the exact cross section formula which, from equation (4.1), can be given as

$$\frac{d\sigma^{(\pm n)}}{d\Omega_f} = \frac{p_f}{p_i} \frac{d\sigma_{\text{Born}}}{d\Omega_f} (\eta \pm n)!\eta! \left| \sum_{k=0}^{\min(\eta \pm n, \eta)} \frac{(-1)^k |\tau|^{2k+n} \exp(-\frac{1}{2}|\tau|^2)}{(n+k)!k![\min(\eta \pm n, \eta) - k]!} \right|^2. \tag{4.13}$$

In the calculation by Osborn, equation (4.11) was used for all values of n. For $n \sim \eta$ this causes an error since the ratio of $|J_\eta(2|\tau|\sqrt{\eta})|$ (approximate) to $(|\tau|^{2\eta}/\eta!)$ (exact) can be estimated to have a lower limit equal to $(|\tau|\sqrt{\eta}/2)^\eta$. In the case of high intensity ($\eta \gg 1$), this expression does not converge, and one has to use the exact expression (4.13).

5. Photon statistics

According to what was said in §§ 2 and 3, the case of circular polarisation is simpler, and therefore for the sake of brevity we shall consider this case only.

As we have seen in equation (2.8), the stationary states of the system 'electron + one circulary polarised mode' can be given as a product of the momentum eigenfunctions of the electron and the $D_\sigma|\eta\rangle$ photon states. If in the solutions given by equation (2.13) we take $\varphi(t_0)$ in the form of the product of a momentum eigenstate and a Fock state, then the photonic part of the full $\varphi(t)$ (or $\psi(t)$) wavefunction becomes

$$D_\sigma|\eta\rangle \equiv |\eta_\sigma\rangle \tag{5.1}$$

as in the stationary case (but $|\eta\rangle$ here is arbitrary and σ is time dependent). The photon states (5.1) can be expanded in the Fock basis of the number operator

$$|\eta_\sigma\rangle = \sum_{r=0}^{\infty} c_r|r\rangle, \qquad a^+a|r\rangle = r|r\rangle, \tag{5.2}$$

where

$$c_r = \langle r|D_\sigma|\eta\rangle. \tag{5.2a}$$

Consider first the case when $|\eta\rangle$ is a number state. Then the c_r coefficients can easily be calculated from equation (4.4) as

$$c_r = \exp(-\tfrac{1}{2}|\sigma|^2) \sum_{k=0}^{\min(r,\eta)} \frac{\sigma^{r-k}(-\sigma^*)^{\eta-k}}{(r-k)!(\eta-k)!k!}. \tag{5.3}$$

$D_\sigma|\eta\rangle$ therefore represents a superposition of Fock states, where the photon number distribution is given by $|c_r|^2$ of (5.3), depending on the electron parameters through σ. As a consequence of the unitarity of D_σ, the sum of the probabilities $|c_r|^2$ is unity, and the probability distribution function is normalised.

Let us compute the expectation value of the photon numbers in the $|\eta_\sigma\rangle$ state:

$$\begin{aligned}\langle r\rangle &= \sum_{r=0}^{\infty} |c_r|^2 r = \langle\eta|D_\sigma^{-1}a^+aD_\sigma|\eta\rangle \\ &= \langle\eta|(a^+ + \sigma^*)(a+\sigma)|\eta\rangle = \eta + |\sigma|^2.\end{aligned} \tag{5.4}$$

The second moment of the distribution can be computed also by taking into account the (2.5a) displacement property

$$\begin{aligned}\langle r^2\rangle &= \sum_{r=0}^{\infty} |c_r|^2 r^2 = \langle\eta|D_\sigma^{-1}(a^+a)^2D_\sigma|\eta\rangle \\ &= \eta^2 + (4\eta+1)|\sigma|^2 + |\sigma|^4.\end{aligned} \tag{5.5}$$

The mean square deviation from equations (5.4) and (5.5) is

$$\langle(\Delta r)^2\rangle = \langle r^2\rangle - \langle r\rangle^2 = 2|\sigma|^2(\eta + \tfrac{1}{2}). \tag{5.6}$$

Formulae (5.3)–(5.6) can be applied to the time-dependent states as well, if $|\eta\rangle$ is the initial number state. The electron–photon interaction smears out the sharp initial distribution $[c_r(t_0) = \delta_{\eta r}]$ into a distribution centred around $\eta + |\sigma|^2$ with a spread given in (5.6). We see also that the distribution is asymmetric around the initial state (otherwise the expectation value should remain η for all times). In the case of an extremely large initial photon number, the arising distribution will have its main contribution around the initial photon number, i.e. the values $|c_{\eta\pm n}|^2$ ($\eta \gg n$) are of special interest. By using the same approximation as the one when passing from equation (4.5) to equation (4.9), we obtain for the probability of the $r = \eta \pm n$ state

$$|c_{\eta\pm n}|^2 \approx J_n^2(2|\sigma|\sqrt{\eta}). \tag{5.7}$$

We note, however, that this is symmetric around η; therefore, for example, a change in the mean photon number cannot be calculated correctly in this approximation. Another interesting property of (5.7) is the existence of induced resonances, i.e. by changing the intensity we can enhance a certain n and suppress the others. The coefficients behave very similarly in the stationary case. (We also note that the square of

the modulus of the coefficients in the Fourier expansion of the solution of the Schrödinger equation with the classical vector potential $\boldsymbol{A} = 2\alpha\sqrt{\eta}(\boldsymbol{\varepsilon}\, e^{i\omega t} + \boldsymbol{\varepsilon}^* e^{-i\omega t})$ is identical to expression (5.7)).

If, in particular, $|\eta\rangle$ is the vacuum state in equation (5.1) then, as we have already mentioned in connection with equation (2.9), $|\eta_\sigma\rangle$ is a coherent state with parameter σ (Glauber 1963a, b):

$$|\eta_\sigma\rangle = D_\sigma|0\rangle = |\sigma\rangle. \tag{5.8}$$

The distribution of photon numbers is a Poisson distribution with the parameter $|\sigma|^2$.

If we use the multiplication rule of the displacment operator (4.3), it can easily be shown that if the initial state $|\eta\rangle$ is a coherent state with the parameter η (where η is now a complex number), then $|\eta_\sigma\rangle$ also remains a coherent state with the parameter $\eta + \sigma$:

$$|\eta_\sigma\rangle = D_\sigma|\eta\rangle = \exp[-\tfrac{1}{2}(\sigma\eta^* - \sigma^*\eta)]|\eta + \sigma\rangle. \tag{5.9}$$

In this case we have the average photon number

$$\langle r\rangle = |\sigma + \eta|^2 \tag{5.10}$$

and the mean square deviation

$$\langle(\Delta r)^2\rangle = |\sigma + \eta|^2. \tag{5.11}$$

According to equations (5.4)–(5.6) and (5.10) and (5.11), the interaction of photons and electrons increases the mean number of photons and smears out the initial distribution. Since $\sigma \leqslant (E/\hbar\omega)^{1/2}\omega_p/\omega$ (E is the electron energy), σ is usually small and the shift in mean photon number is negligible. However, the mean square deviation as given by (5.6) is proportional to the initial photon number, so that the spread of the developing distribution can be quite wide if $|\eta|$ is large. Essentially the same conclusions can be drawn for the case when the initial state is a coherent state.

Finally we note that if σ is time dependent, then the average value, the mean square deviation, etc of the distribution generated by $|\eta_\sigma\rangle$ oscillates in time with an amplitude proportional to $|e\alpha\boldsymbol{p}\boldsymbol{\varepsilon}/mc\hbar\Omega|^2$. The time average of these oscillations is of the order $(E/\hbar\omega)(\omega_p/\omega)^2$ and is therefore negligible in comparison with unity.

6. Summary and discussion

We have already pointed out in the Introduction that, for the description of the scattering processes of a free electron in the presence of an intense field, semiclassical methods are not always applicable. On the other hand, it is also true that perturbative treatment of the high-order nonlinear processes using a fully quantised description is extremely clumsy. In this paper we have developed a relatively simple method to handle such problems. In § 4 we have illustrated for the nonlinear inverse bremsstrahlung how one can determine in one step the amplitude of transitions between arbitrarily 'distant' excitation levels of one quantised radiation mode induced by the scattering of the electron. To perform this, in §§ 2 and 3 we have exactly solved the Schrödinger equation of the system 'electron + one quantised EM mode' in dipole approximation for linear and circular polarisations. With the help of the solutions (2.8*b*) we have calculated analytically the cross section of the direct and inverse bremsstrahlung of an electron scattered by a background potential (equations (4.11), (4.12) and (4.13)). We

have shown that the approximation (4.11) for the case of large initial photon numbers, which is analogous to the semiclassical result, can be very different from the exact result when applied to the depletion of the mode. With this in mind, we cannot completely agree with the pure energy absorption coefficient of a plasma as calculated by Osborn (1972), since he seems to have overlooked this discrepancy.

In both cases (linear and circular polarisation) we have found that the spectrum of the system 'electron + one quantised mode' belonging to the stationary states parametrised by the quantum numbers $\boldsymbol{p}$ and η consists of essentially two parts, namely one continuous part $(\boldsymbol{p}^2/2m)$ and one discrete part $(n\hbar\Omega$ or $n\hbar\nu)$, as shown by equations (2.8*c*) and (3.6*b*). Ω and ν denote renormalised frequencies shifted towards higher frequencies with respect to the original ω frequency.

We also mention that the above Schrödinger equation can be solved almost exactly beyond dipole approximation as well, by using our method, if one introduces a new parameter $\sigma' = \sigma\, e^{-ikr}$ in the D transformation. The main calculational difficulty is caused in this case by the recoil.

From the results of § 3 we exactly determined the energy correction arising from the $\boldsymbol{A}^2$ term for linear polarisation. The main contribution is due to the $\Delta\omega = \Omega - \omega$ shift and is usually very small. The solutions (3.6*a*), according to (3.7) and (3.7*a*, *b*, *c*), are superpositions of states where the influence of the $\boldsymbol{A}^2$ terms is taken into account up to infinite order.

In § 5 we dealt with the photon number distribution generated by the solutions given in the previous sections. Since the expectation values of the energies of the electron and the radiation mode are $(\boldsymbol{p}^2/2m)$ and $\hbar\omega(n + \frac{1}{2} + |\sigma|^2)$, the interaction energy $-\hbar\Delta\omega(n + \frac{1}{2} + |\sigma|^2) - 2\hbar\omega|\sigma|^2$ appearing in the total energy of the system according to (1.8*c*) is very small in comparison with the energies of the subsystems (electron versus photons), and the photons are only weakly coupled to the electron. This does not hold, however, when the full radiation field (without restriction to one intense mode) is considered. But in the one-mode approximation this can also be seen from the fact that the expectation value of the mode energy as given by (5.4) is shifted only very slightly due to the interaction with the electron. The general time-dependent solutions obtained in § 2 represent coherent states if the initial state of the mode was a coherent state (the vacuum in particular). In both cases (coherent or number initial state), the expectation values of the photon number are shifted and the mean square deviations increase, i.e. interaction with the electron smears out the distribution function. Since the field developing from the vacuum is the self field of the electron, on the basis of our exact results we may state that the low-frequency part of the self field of the electron is coherent or, in other words, is a classical field. The extension of the above results to the relativistic case and beyond dipole approximation (Dirac electron + a quantised EM mode) is in progress and will be published in a separate paper where the power of the method will be demonstrated on other scattering processes such as nonlinear Compton scattering.

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